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On the Relation Between Time Representations and Inner Product Spaces

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Stable states (particles), ghosts and unstable states (particles) come with different types of time representations in unitary groups—definite or indefinite. These representations are discussed with respect to the induced inner product spaces as extensions of Hilbert spaces. Unstable particles with their decay channels are treated as higher dimensional probability collectives.

KEY WORDS: probability; indefinite metric; unstable particles.

1. INTRODUCTION

As codified by J. von Neumann, quantum mechanics is formulated in a Hilbert space, and this by very good physical and mathematical reasons: First of all, Hilbert space scalar products are appropriate to obtain experimentally testable probabilities from a quantum mechanical theory which replace the deterministic numbers of classical theories. Mathematically, Hilbert spaces share many simple properties with finite dimensional spaces, e.g., isomorphisms to the dual space or endomorphism[2] C^*-algebras.

However, there are important places where a Hilbert space formulation seems too narrow to accommodate other important physical structures, especially if quantum theory is based on an operationally motivated group theoretical approach. It is the purpose of this talk to review and to discuss—with some of the unsolved problems—two extensions of Hilbert spaces—the case of gauge theories and the case of unstable particles.

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[2] The linear mappings of a vector space $\{V \to V\}$ are its endomorphisms, they constitute a unital associative algebra $\mathbf{AL}(V)$. The regular group $\mathbf{GL}(V)$ therein are the V-automorphisms, i.e., the invertible endomorphisms.

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In quantum mechanics the irreducible definite unitary[3] representations of position translations (characters) $\vec{x} \to e^{i\vec{q}\vec{x}} \in \mathbf{U}(1)$ are not square integrable. With the well-known representation theory of the additive groups $\mathbb{R}^s$ (Boener, 1955) and the related harmonic analysis (Fourier transformation) (Folland, 1995) it may seem awkward that the nondecomposable, even the irreducible position translation $\vec{x} \in \mathbb{R}^3$ representations, used for scattering states, do not constitute vectors in the Hilbert space of the square integrable position space functions (Schrödinger picture). To obtain Hilbert space vectors, the translation eigenstates have to come with square integrable momentum wave packets. One way to reconcile space translation eigenvectors with Hilbert space structures—not discussed in the following—is to give up the selfduality of the position representation space worked with and to sandwich a Hilbert space between a dual pair of topological vector spaces in a Gel'fand space triplet (rigged Hilbert space)(Gel'fand and Vilenkin, 1964). In such an extended theory the probability interpretation for experiments, e.g., a substitute for the Parseval equation (decomposition of the unit) in Hilbert spaces, has to be found, discussed or abolished, a question which to my knowledge is not solved satisfactorily yet.

Another important place where operations enforce an extension of Hilbert space structures are Lorentz transformation compatible theories with massless Lorentz vector fields as classically first used with the vector potential in electrodynamics and later in the framework of non-Abelian gauge theories. There arises a clash between the causality compatible indefinite Lorentz metric for the four spacetime translations and its use as metric for state vectors related to the four components of a Lorenz vector field. For a massive vector field, e.g., a weak neutral Z-boson (idealized as stable), a rest system projection is compatible with the definite subgroup projection from the orthochronous Lorentz group to the spin group $\mathbf{SO}_0(1, 3) \to \mathbf{SO}(3)$; therewith a quantum field of the massive particle can come with the three spin components only. In contrast to the massive case, a massless vector field needs more than its two polarized particle degrees of freedom. The additional two degrees of freedom in the four component massless Lorentz vector field are the gauge degree of freedom and—in an electrodynamical language—the Coulomb degree of freedom. These two nonparticle degrees of freedom reflect the projection of the indefinite orthogonal group $\mathbf{SO}_0(1, 3) \to \mathrm{SO}_0(1, 1)\times \mathbf{SO}(2)$ involving the axial rotations as little group for the polarization of the massless particles. In the complex vector spaces of quantum theory, particles come with definite unitary groups and the related Hilbert spaces, in the vector field example with the embedding $\mathbf{SO}(3) \hookrightarrow \mathbf{U}(3)$ for the three massive spinning vector field components and $\mathbf{SO}(2) \hookrightarrow \mathrm{U}(2)$ for the two massless polarized vector field components. In the latter case, the massless nonparticle degrees of

[3] In this paper, "unitary" includes both "definite and indefinite unitary." Orthogonal and unitary groups can come with a nontrivial signature $\mathbf{O}(n_+,n_-)$ and $\mathbf{U}(n_+,n_-)$ which for $n_+n_- > 0$ are called indefinite.

freedom are related to an indefinite unitary group $\mathbf{SO}_0(1,1) \hookrightarrow \mathbf{U}(1,1)$ acting upon a "non-Hilbert space." In the context of massless vector field theories, a probability compatible coexistence of indefinite unitary inner product spaces, used for a local formulation of the related gauge interaction, and definite unitary Hilbert spaces for free noninteracting particles is solved. Its algebraic formulation relying on the structure of reducible, but nondecomposable time representations(Boener, 1955) and the connection between the quantum appropriate nilpotent Jordan transformations (BRS-transformations; Becchi *et al.*, 1976) and the only classically appropriate gauge transformations (with spacetime dependent Lie parameters) has been clarified (Becchi, *et al.*, 1976; Kugo and Qjina, 1978) and will be shortly reviewed in the section "Ghost without Particles" using the language of indefinite unitary inner product spaces.

For unstable particles, indefinite unitary groups are unavoidable as illustrated, e.g., by the representation matrix element $\mathbb{R} \ni \mapsto e^{i(E+i\Gamma/2)t}$ involving a complex time translation eigenvalue $iE - \Gamma/2$ with real energy E and positive width Γ. Since the Hilbert space scalar product for the state vectors and particles (translation eigenstates) is induced by the unitary group which contains the time translation representation, as sketched in the section "Stable Particles and States," the arising problems for an indefinite unitary group, appropriate for unstable particles, are apparent. In another contribution to this conference (Blum and Saller, in press), we have proposed to treat the indefinite unitary structure arising for unstable particles in a relativistic treatment as a position translation representation related phenomenon by considering the width Γ as an invariant related to a full-fledged Lorentz vector. In such a proposal there exist "sharp energy Lorentz systems" for unstable particles where the time translations are represented in a definite unitary group and, therewith, may induce—as for stable particles—a Hilbert space structure where their indefinite unitary position translation representation is considered in. As well known from quantum mechanics, indefinite unitary representation matrix elements arise for position translations, e.g. the knotless wave functions $r \to e^{-\frac{r}{k}} \notin \mathbf{U}(1)$ (principal quantum number $k = 1, 2, \ldots$) in the nonrelativistic hydrogen atom. In such an approach, instability and indefinite unitary structure originate ultimately from the position degrees of freedom, not from the time degree of freedom. Much additional work has to be done to clarify the viability of this proposal.

In addition to and connected with this definite–indefinite unitary features in the time and position translation representations of unstable particles, there arises the phenomenon of "probability collectives": An unstable particle has to be treated together with its decay products and possibly, as familiar from the neutral kaon system, together with other unstable partner particles. Such particle collectives, shortly presented in the section "Unstable States and Particles," constitute Hilbert subspaces with positive metrices, nondiagonalizable in the particle basis. They are higher dimensional generalizations of the 1-dimensional Hilbert subspace lattices for stable state vectors with the logical framework as formalized by Birkhoff and von Neumann (1936). Questions like the probability normalization of unstable particle collectives, transition amplitudes, etc. have to be discussed (Saller, in press) which will only be alluded to.

2. DYNAMICS, UNITARITY AND TIME REFLECTION

In quantum theory, a physical dynamics contains a complex representation of the real time (translations). It comes with a conjugation which defines an inner product. A definite unitary product can be used for transition elements and probabilities. In this section, the general mathematical structures for the connection between time representation groups in complex vector space automorphisms and the induced inner products is shortly reviewed (Bourbaki, 1959).

As model for time the Abelian totally ordered real numbers are used—either multiplicatively, called time group

$$\mathbf{D}(1) = \{e^t | t \in \mathbb{R}\}$$

or additively, called time Lie algebra[4] with the time translations

$$\mathbb{R} = \log\mathbf{D}(1) = \{t\}$$

Obviously, as Lie groups $(\mathbf{D}(1), \cdot)$ and $(\mathbb{R}, +)$ as isomorphic.

As familiar from the simplest case, the harmonic oscillator $t \to e^{iEt}$, a quantum dynamics is a representation of time as group in the automorphisms $\mathbf{GL}(V)$ and as Lie algebra in the endomorphisms $\mathbf{AL}(V)$ of a complex vector space V

$$\text{group}: \quad D : \mathbf{D}(1) \to \mathbf{GL}(V), e^t \to D(t), \begin{cases} D(0) &= id_V \\ D(t+s) &= D(t) \circ D(s) \end{cases}$$

$$\text{Lie algebra}: \quad \mathcal{D} : \mathbb{R} \to \mathbf{AL}(V), t \to \mathcal{D}(t) = t\mathcal{D}(1)$$

The represented basis of the time translations is—up to i—the Hamiltonian

$$\mathcal{D}(1) = iH$$

For the harmonic oscillator one has a 1-dimensional representation space $V \cong \mathbb{C}$ with $\mathbf{GL}(\mathbb{C})$ the nontrivial numbers and $\mathbf{AL}(\mathbb{C})$ all complex numbers. The harmonic oscillator Hamiltonian is the frequency E in $\mathcal{D}(1) = iE$.

In general, a solution of a dynamics is a triagonalization of the Hamiltonian—perhaps even a diagonalization in the case of only irreducible representations. Reducible, but nondecomposable contributions cannot be spanned by eigenvectors. States (bound states, scattering states, particles) have to be eigenvectors under time action.

[4] The Lie algebra for a Lie group G will be denoted by $\log G$.

The complex representation space $V \cong \mathbb{C}^n$ (finite dimensional for a simple formulation of the concepts used) has to come with a conjugation in order to represent the realness of the Lie structure time with concepts like hermitian (real), anti-Hermitian (imaginary), etc. For the harmonic oscillator $V \cong \mathbb{C}$, this conjugation is the canonical complex number conjugation $\alpha \leftrightarrow \overline{\alpha}$. In general, a conjugation of a complex vector space V is an antilinear isomorphism to its dual space V^T (vector space with the linear V-forms)

$$V \overset{*}{\leftrightarrow} V^T, \quad v \leftrightarrow v^*, \quad v^{**} = v$$
$$\text{Dirac notation} : v = |v\rangle, \quad v^* = \langle v|$$

For spaces with dimension $n\ N \geq 2$ there is not only the conjugation induced by the canonical number conjugation.

There may exist even more than one physically relevant conjugation for one vector space, e.g., in particle physics, the conjugation connecting creation and annihilation operators u $\leftrightarrow$ u*, a $\leftrightarrow$ a* together with the conjugation connecting particle creation with antiparticle annihilation and vice versa u $\leftrightarrow$ a*, a $\leftrightarrow$ u*.

With a conjugation the V-endomorphisms $f \in \mathbf{AL}(V)$ can be conjugated too by using the transposed endomorphisms $f^T : V^T \to V^T$

$$\mathbf{AL}(V) \overset{*}{\leftrightarrow} \mathbf{AL}(V) \text{ with } [f : V \to V]$$
$$\overset{*}{\leftrightarrow} [f^* : V \overset{*}{\to} V^T \overset{f^T}{\to} V^T \overset{*}{\to} V]$$

The realness of the time group $t = \overline{t} \in \mathbb{R}$ is implemented by the hermiticity of the Hamiltonian—with respect to the conjugation $*$

$$H = H^*$$

generalizing the realness $E = \overline{E}$ of the harmonic oscillator frequency. Therewith, the generator $iH = \mathcal{D}(1)$ for the represented time translations is anti-Hermitian and the represented time group unitary $e^{iHt} = D(t)$—always with respect to the conjugation $*$

$$\mathcal{D}(1)^* = \mathcal{D}(-1) = -\mathcal{D}(1) \quad * -\text{anti-Hermitian for time translations}$$
$$\mathcal{D}(t)^* = \mathcal{D}(-t) = -\mathcal{D}(t)^{-1} \quad * -\text{unitary for group representation}$$

The conjugation of the complex representation space V together with the action of the represented time translations allows the implementation of the time reflection by endomorphism conjugation. Time reflection interchanges with future and past also bra and ket which—for scattering processes—are used for incoming and outgoing states

$$\begin{array}{ccc} \underset{\displaystyle D(t), \mathcal{D}(t)}{\overline{t}\downarrow} & \mapsto & \underset{\displaystyle D(t)^*, \mathcal{D}(t)^*}{\overline{-t}\downarrow} \end{array} , \quad \begin{array}{c} |v\rangle \overset{*}{\leftrightarrow} \langle v| \\ \langle v|w\rangle \overset{*}{\leftrightarrow} \langle w|v\rangle \\ \langle v|v \in \mathbb{R} \end{array}$$

With the dual product (bilinear) for the vector space V and its linear forms

$$V^T \times V \to \mathbb{C}, \quad (\omega, w) \mapsto \langle \omega, w \rangle$$

a conjugation is equivalent to an inner product (a nonsingular sesquilinear form)

$$V \times V \to \mathbb{C}, \quad (v, w) \mapsto \langle v|w\rangle = \langle v^*, w\rangle$$

The unitary group, characterizing and equivalent to the conjugation $*$, is the invariance group of the induced inner product

$$\mathbf{U}(V, *) = \{u \in \mathbf{GL}(V) | \langle u.v | u.w \rangle = \langle v|w\rangle \quad \text{for all} \quad v, w \in V\}$$

This defines the signature of the unitarity and of the conjugation

$$V \cong \mathbb{C}^n, \langle\,|\,\rangle \cong \begin{pmatrix} \mathbf{1}_{n+} & 0 \\ 0 & -\mathbf{1}_{n-} \end{pmatrix},$$
$$n_+ + n_- = n, \quad \mathbf{U}(V, *) = \mathbf{U}(n_+, n_-)$$

There are as many different types of conjugations as there are signatures—for $n = 1$ only $\mathbf{U}(1)$. An indefinite unitary group needs more than one dimension. for $n = 2$ one has two types: $\mathbf{U}(2)$- and $\mathbf{U}$(1,1)-conjugations, etc. With the exception of the Euclidean conjugations $\mathbf{U}$(n), denoted with the five-cornered star $*$, where one has a scalar product and a Hilbert space structure for the vector space V, all conjugations and associated inner products are indefinite. In the following time representations with both definite and indefinite conjugations will be considered.

To become familiar with indefinite conjugations a $\mathbf{U}$(1,1) inner product is considered which will be used in section "Ghosts Without Particles"

$$\text{for} \quad \mathbf{U}(1, 1) : \langle\,|\,\rangle \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Us usual, the inner product matrix depends on the vector space basis: A diagonal matrix with the explicit signature arises for Sylvester bases(Bourbaki, 1959) whereas neutral, not orthogonal pairs with trivial norm show up in Witt bases (Bourbaki, 1959). In the example above Sylvester and Witt bases are related to each other by the automorphism $w = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The inner product defines a $\mathbf{U}$(1,1)-conjugation, denoted with $\times$, which—in the basis with the skew-diagonal matrix for the inner product—interchanges the conjugated diagonal elements in contrast to the more familiar definite $\mathbf{U}$(2)-conjugation $*$ (number conjugation and transposition) which interchanges the conjugated skew-diagonal elements

$$\mathbf{U}2 = \star : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{\star} = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix}$$

$$\mathbf{U}(1,1) = \times : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{\times} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{\star} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\delta} & \overline{\beta} \\ \overline{\gamma} & \overline{\alpha} \end{pmatrix}$$

Obviously, **U**(2) and **U**(1,1)-hermiticity or unitarity are different.

The product of two conjugations gives a vector space automorphism, e.g., for **U**(1,1) and **U**(2)-conjugation $\times \circ \star = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The basis dependence of inner product matrices is given by the symmetric space $\mathbf{GL}\,(\mathbb{C}^n)/\mathbf{U}(n_+, n_-)$, i.e. the orientation manifold of a unitary group $\mathbf{U}n_+, n_-$, e.g., $\mathbf{GL}\mathbb{C}/\mathbf{U}(1) \cong \exp \mathbb{R}$ for 1-dimension, leaving a free normalization, or the real 4-dimensional inner product manifolds $\mathbf{GL}\,(\mathbb{C}^2)/\mathbf{U}(2)$ and $\mathbf{GL}\,(\mathbb{C}^2)/\mathbf{U}(1,1)$ for two dimensions.

3. THREE CHARACTERISTIC TIME REPRESENTATIONS

As discussed above, complex representations of the real-time group **D**(1) and time translations $\mathbb{R}$ come in unitary groups and their Lie algebras. The relevance of the three representation properties "*unitary*," "*faithful*" (*injective*), and "*irreducible*" for the induced inner products will be discussed in the following. After a first orientation in this section, three characteristic representation types (Boerner, 1955; Saller, 1989) will be treated in more detail below with respect to the probability interpretation.

Irreducible unitary representations of time with implemented time reflection are definite unitary in **U**(1)

$$\mathbf{D}(1) \ni e^t \mapsto e^{iEt} \in \mathbf{U}(1) = \exp i\mathbb{R}$$
$$\mathbb{R} \ni t \mapsto iEt \in \log \mathbf{U}(1) = i\mathbb{R}$$

with eigenvalue $iE \in i\mathbb{R}$

These time representations are not faithful: With the periodicity of e^{iEt} the image of of the noncompact group **D**(1) (line) is its compact quotient group **U**(1) (1-torus). They are used for stable states (particles). As well known and repeated in Section 4, the definite unitary group induces Hilbert space structures with the quantum characteristic probability amplitudes.

The smallest *unitary* and *faithful* time representations are indefinite unitary, they are in **U**(1,1)

$$\mathbf{D}(1) \ni e^t \mapsto e^{iEt} \begin{pmatrix} 1 & i\nu t \\ 0 & 1 \end{pmatrix} \quad \in \mathbf{U}(1,1), \quad 0 \neq \nu \in \mathbb{R}$$

$$\mathbb{R} \ni t \mapsto i \begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix} t \quad \in \log \mathbf{U}(1,1)$$

with 2×2-Hamilton-matrix : $H = \begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix}$

eigenvalue $iE \in i\mathbb{R}$

These reducible, but nondecomposable time representations can be written with triangular Jordan matrices involving a basis dependent nontrivial nilconstant ν. There is a 1-dimensional time translation invariant subspace in the 2-dimensional representation space. However, there in no invariant complementary space. This will be illustrated in Section 5.1. Such time representations are used in quantum field theories to describe ghosts, i.e. for interaction degrees of freedom without asymptotic particle states, e.g. for the nonphotonic degrees of freedom in gauge fields, i.e. the Coulomb force and the gauge degrees of freedom where the nilconstant is the gauge fixing constant (Section 5).

Irreducible faithful time representations are not unitary. They have no time reflection

$$e^t \mapsto e^{i\left(E+i\frac{\Gamma}{2}\right)^t} \quad \in \mathbf{GL}(\mathbb{C}) = \mathbf{U}(1) \times \mathbf{D}(1),\ \Gamma > 0$$

$$t \mapsto i\left(E + i\frac{\Gamma}{2}\right)^t \quad \in \log \mathbf{GL}(\mathbb{C}) = \mathbb{C}$$

with eigenvalue $\quad iE - \dfrac{\Gamma}{2} \in \mathbb{C}$

The representations of the future cone $t \geq 0$ with the eigenvalues having a nontrivial real part (width) are used for decaying states, e.g. for unstable particles (Section 6). The corresponding decomposable indefinite unitary representations with conjugation implemented time reflection involve two reflected eigenvalues $iE \mp \frac{\Gamma}{2}$

$$\mathbf{D}(1) \ni e^t \mapsto e^{iEt} \begin{pmatrix} e^{-\frac{\Gamma}{2}t} & 0 \\ 0 & e^{+\frac{\Gamma}{2}t} \end{pmatrix}$$

$$\sim e^{iEt} \begin{pmatrix} \cosh\frac{\Gamma}{2}t & \sinh\frac{\Gamma}{2}t \\ \sinh\frac{\Gamma}{2}t & \cosh\frac{\Gamma}{2}t \end{pmatrix} \in \mathbf{U}(1,1)$$

$$\mathbb{R} \ni t \mapsto \begin{pmatrix} i\left(E + i\frac{\Gamma}{2}\right) & 0 \\ 0 & i\left(E - i\frac{\Gamma}{2}\right) \end{pmatrix} t$$

$$\sim \begin{pmatrix} iE & \frac{\Gamma}{2} \\ \frac{\Gamma}{2} & iE \end{pmatrix} t \in \log \mathbf{U}(1,1)$$

4. STABLE STATES AND PARTICLES

Quantum probability as used with the Hilbert space formulation of quantum mechanics is induced by irreducible time representations in **U**(1). The well-known construction from **U**(1)-conjugation to Fock–Hilbert space is shortly reviewed. Ultimately it is used for the quantum treatment of all stable states and particles.

The harmonic Bose and Fermi oscillator are the quantum representations of the irreducible time representation, starting from the representations

$$t \mapsto \pm iEt, \quad e^t \mapsto e^{\pm iEt} \text{ with } E \in \mathbb{R}$$

on 1-dimensional dual vector spaces with dual bases (u, u⋆). These dual bases give rise (below) to the creation and annihilation operator. They are related to each other by the **U**(1)-conjugation ⋆

$$\mathbf{U}(1) : V = \mathbb{C}u \overset{\star}{\leftrightarrow} V^T = \mathbb{C}u^*$$
$$V \times V \to \mathbb{C}, \langle u|u\rangle = \langle u^*, u\rangle = 1$$

The time translation generator is the basic space identity $iEid_V$ which can be written as tensor product of the dual basic vectors (creation-annhilation operators)

$$iH = iE\text{id}_V = iEu \otimes u^\star = iE|u\rangle\langle u|$$

The quantum implementation of the irreducible time representations in **U**(1) uses the noncommutative quantum algebras (Saller, 1993b,c) of the direct sum space $V \oplus V^T \cong \mathbb{C}^2$. Quantum algebras are duality induced quotient algebras of the multilinear tensor algebra of $V \oplus V^T$ and contain as elements all complex creation-annihilation polynomials ℂ[u,u⋆], modulo the duality induced (anti) commutators with the notation $[a, b]_\epsilon = ab + \in ba$. The quantum algebra is finite dimensional for the Fermi case (Pauli principle) with basic anticommutators and countably infinite dimensional for the Bose case with basic commutators

$$\left.\begin{array}{l}\epsilon = +1 \text{ Fermi}\\ \epsilon = -1 \text{ Bose}\end{array}\right\} : \mathbf{Q}_\epsilon(\mathbb{C}^2) = \mathbb{C}[\text{u}, \text{u}^\star]/ \quad \text{modulo} \quad \left\{\begin{array}{l}[\text{u}^\star, \text{u}]_\epsilon - 1\\ \quad [\text{u}, \text{u}]_{\epsilon,} \quad [\text{u}^\star, \text{u}^\star]_\epsilon\end{array}\right.$$

$$\dim \mathbf{Q}_\epsilon(\mathbb{C}^2) = \begin{cases} 4, & (\text{Fermi})\\ \aleph_0, & (\text{Bose})\end{cases}$$

In the quantum algebras the Hamiltonian for the **U**(1)-time representation above can be written as the quantization opposite commutator. Its adjoint action implements the time translations which are formulated by the equations of motion

$\frac{da}{dt} = \mathrm{ad}\, i\mathbf{H}(a) = [i\mathbf{H}, a]$

$$\mathbf{H} = E\frac{[\mathrm{u}, \mathrm{u}^\star]_{-\epsilon}}{2} \in \mathbf{Q}_\epsilon(\mathbb{C}^2) \Longrightarrow \begin{cases} [i\mathbf{H}, \mathrm{u}] & = iE\mathrm{u} \\ [i\mathbf{H}, \mathrm{u}^\star] & = -iE\mathrm{u}^\star \end{cases}$$

The Hilbert spaces of quantum mechanics are quotient structures of the quantum algebras: An inner product is constructed for the quantum algebra by the canonical extension of the scalar product $\langle \mathrm{u}^\star, \mathrm{u}\rangle = 1$ for the conjugation group $\mathbf{U}(1)$ on the basic vector space V

$$\mathbf{Q}_\epsilon(\mathbb{C}^2) \times \mathbf{Q}_\epsilon(\mathbb{C}^2) \to \mathbb{C},$$

$$\langle a|b\rangle = \angle a^\star b \text{ with } \begin{cases} \angle(\mathrm{u}^\star \mathrm{u})^k & = \angle \mathrm{u}^\star \mathrm{u}^k = 1 \\ \text{for } k & = 0, 1, \ldots \\ \angle(\mathrm{u}^\star)^k (u)^l & = 0 \text{ for } k \neq l \end{cases}$$

The inner product is positive semidefinite $\langle a|a\rangle \geq 0$ i.e. the quantum algebra is a pre-Hilbert space. With the annihilation left ideal in the quantum algebra

$$\{n \in \mathbf{Q}_\epsilon(\mathbb{C}^2) | \langle a|n\rangle = 0 \text{ for all } a \in \mathbf{Q}_\epsilon(\mathbb{C}^2)\} = \mathbf{Q}_\epsilon(\mathbb{C}^2)\mathrm{u}^\star$$

one defines the corresponding equivalence classes

$$\mathbf{Q}_\epsilon(\mathbb{C}^2) \to \mathrm{FOCK}_\epsilon(\mathbb{C}^2), \quad a \mapsto |a\rangle = a + \mathbf{Q}_\epsilon(\mathbb{C}^2)\mathrm{u}^\star$$

They constitute a complex vector space with definite scalar product, the Fock space. It is 2-dimensional for the Fermi quantum algebra and $\aleph_0$-dimensional for the Bose quantum algebra

$$\mathrm{FOCK}_\epsilon(\mathbb{C}^2) = \mathbf{Q}_\epsilon(\mathbb{C}^2)/\mathbf{Q}_\epsilon(\mathbb{C}^2)\mathrm{u}^\star \cong \begin{cases} \mathbb{C}^2, & \text{Fermi} \\ \mathbb{C}^{\aleph_0}, & \text{Bose} \end{cases}$$

$$\mathrm{FOCK}_\epsilon(\mathbb{C}^2) \times \mathrm{FOCK}_\epsilon(\mathbb{C}^2), \quad \langle a\|b\rangle = \angle a^\star b,\ \langle a\|a\rangle = 0 \iff |a\rangle = 0$$

The Fock space can be spanned by the energy eigenvectors of the $\mathbf{U}(1)$-time action as implemented in the quantum algebra

$$\text{basis of } \mathrm{FOCK}_\epsilon(\mathbb{C}^2) : \{|k\rangle = \frac{(u)^k}{\sqrt{k!}} + \mathbf{Q}_\epsilon(\mathbb{C}^2)\mathrm{u}^\star | \begin{matrix} k = 0, 1\} & \text{Fermi} \\ k = 0, 1, 2, \ldots\} & \text{Bose} \end{matrix}$$

$$\mathrm{FOCK}_\epsilon(\mathbb{C}^2) \times \mathrm{FOCK}_\epsilon(\mathbb{C}^2) \mapsto \mathbb{C},\ \langle k\|l\rangle = \delta_{kl}$$

Its Cauchy completion defines the Hilbert spaces, Fermi or Bose, associated to an irreducible unitary time representation.

The position representation—only possible for the Bose case

$$\text{for } \mathbf{Q}_-(\mathbb{C}^2) : \quad x = \frac{\mathrm{u}^\star + \mathrm{u}}{\sqrt{2}}, \quad ip = \frac{d}{dx} = \frac{\mathrm{u}^\star - \mathrm{u}}{\sqrt{2}}, \quad \mathbf{H} = E\frac{\{\mathrm{u}, \mathrm{u}^*\}}{2}$$

$$= E\frac{p^2 + x^2}{2}$$

gives the scalar product Fock space in the form of the rapidly decreasing functions with the square integrable functions as their scalar product norm completion[5]

$$\mathbf{Q}_-(\mathbb{C}^2)/\mathbf{Q}_-(\mathbb{C}^2)\mathrm{u}^\star = \mathrm{FOCK}_-(\mathbb{C}^2) \cong S(\mathbb{R}) \subset L^2_{dx}(\mathbb{R}) \cong \overline{\mathrm{FOCK}_-}(\mathbb{C}^2)$$

The degrees of the Hermite polynomials $\deg \mathrm{H}^k = k$ in a (Hilbert) basis reflect the $\mathbb{Z}$-grading of the Bose quantum algebra $\mathbf{Q}_-(\mathbb{C}^2)$

$$\frac{\mathrm{u}^k}{\sqrt{k!}}|0\rangle = |k\rangle \cong \psi^k : \begin{cases} \mathrm{u}^\star|0\rangle &= 0 \Longrightarrow (x + \frac{d}{dx})\psi_0(x) = 0 \\ \psi^k(x) &= \frac{1}{\sqrt{ki}}\left(\frac{x - \frac{d}{dx}}{\sqrt{2}}\right)\psi_0(x) \\ &= \frac{1}{\sqrt{2^k k!\sqrt{\pi}}} e^{\frac{x^2}{2}}(-\frac{d}{dx})^k e^{-x^2} \\ &= \frac{1}{\sqrt{2^k k!\sqrt{\pi}}} e^{-\frac{x^2}{2}}\mathbf{H}^k(x) \end{cases}$$

In addition to the time translation orbits, e.g., $t \mapsto e^{ikEt|k\rangle}$ and the scalar product for transition elements and probabilities, e.g., $\langle k|l\rangle$, Schrödinger wavefunctions are also position translation orbits

$$\mathbb{R} \oplus \mathbb{R} \ni (t, x) \mapsto e^{ikEt}\psi^k(x) \in \mathbf{U}(1) \times L^2_{dx}(\mathbb{R})$$

and introduce a position spread for amplitudes and probabilities.

All particle quantum fields are built with harmonic oscillators where the creation and annihilation operators are indexed with continuous momenta $\vec{q} \in \mathbb{R}^3$ for $q_0 = \sqrt{m^2 + q^2}$

$$V_{\vec{q}} = \mathbb{C}\mathrm{u}(\vec{q}) \overset{\star}{\leftrightarrow} V^T_{\vec{q}} = \mathbb{C}\mathrm{u}^\star(\vec{q})\,\text{with} \quad \begin{cases} [\mathrm{u}^\star(\vec{p}), \mathrm{u}(\vec{q})]_\epsilon &= (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \\ [\mathrm{u}(\vec{p}), \mathrm{u}(\vec{q})]_\epsilon &= 0 \\ [\mathrm{u}^\star(\vec{p}), \mathrm{u}^\star(\vec{q})]_\epsilon &= 0 \\ \angle \mathrm{u}^\star(\vec{p})\mathrm{u}(\vec{q}) &= (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \end{cases}$$

e.g. for a stable spinless π^0 with a Lorentz scalar field Φ, for a stable spin 1 Z^0 with a Lorentz vector field Z or for the spin $\frac{1}{2}$ electron–positron with particles (u, u*) and antiparticles (a,a*) in the left and right handed contributions with a Dirac field $\Psi = (\mathbf{r}^A, \mathbf{1}^{\dot{A}})$—all given by direct integrals $\int^\oplus$ over the boost degrees of freedom

$$\Phi(x) = \int^\oplus \frac{d^3q}{(2\pi)^3 q_0} \frac{u(\vec{q})e^{iqx} + \mathrm{u}^\star(\vec{q})e^{-iqx}}{\sqrt{2}}$$

$$\mathbf{Z}^j(x) = \int^\oplus \frac{d^3q}{(2\pi)^3 q_0} \Lambda\left(\frac{q}{m}\right)^j_a \frac{u^a(\vec{q})e^{iqx} + \mathrm{u}^{\star a}(\vec{q})e^{-iqx}}{\sqrt{2}}, \begin{cases} j = 0, 1, 2, 3 \\ a = 1, 2, 3 \end{cases}$$

[5] At this point, and only for Bose structures, one may dualize to Gel'fand triplets, here $S(\mathbb{R}) \subset L^2_{dx}(\mathbb{R}) \cong [L^2_{dx}(\mathbb{R})]' \subset S(\mathbb{R})'$ with the tempered distributions.

$$\mathbf{r}^A(x) = \int^{\oplus} \frac{d^3q}{(2\pi)^3 q_0} s\left(\frac{q}{m}\right)^A_\alpha \frac{u^\alpha(\vec{q})e^{iqx} + a^{\star\alpha}(\vec{q})e^{-iqx}}{\sqrt{2}}$$

$$\mathbf{1}^{\dot{A}}(x) = \int^{\oplus} \frac{d^3q}{(2\pi)^3 q_0} s^{-1}\left(\frac{q}{m}\right)^{\dot{A}}_\alpha \frac{u^\alpha(\vec{q})e^{iqx} - a^{\star\alpha}(\vec{q})e^{-iqx}}{\sqrt{2}}, \begin{cases} A, \dot{A} = 1, 2 \\ \alpha \quad\; = 1, 2 \end{cases}$$

In addition to the momentum measure, the nonscalar fields involve the corresponding representations of the boosts $\mathbf{SO}_0(1,3)/\mathbf{SO}(3) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$—here the Weyl and vector representations $\{s(\frac{q}{m}), \Lambda(\frac{q}{m})\}$.

5. GHOSTS WITHOUT PARTICLES

Unitary and faithful time representation are in the indefinite unitary group $\mathbf{U}(1,1)$. Such reducible, but nondecomposable representations are used for the nonparticle ghost degrees of freedom[6] in gauge and Fadeev–Popov fields. Historically, gauge fields were introduced in order to "compatibilize" space–time translations with space–time-dependent Lie group transformations. The probability problems in quantum electrodynamics, arising from the indefinite state space metric, induced by the indefinite Lorentz metric for space–time translations, the subgroup $\mathbf{O}(1,1) \subset \mathbf{O}(1,3)$ were exacerbated in Yang–Mills theories and finally solved (Kugo and Djuina, 1978) by replacing the space–time-dependent gauge group transformations by nilpotent BRS-transformations (Becchi *et al.*, 1976), involving additional nonparticle Fadeev–Popov degrees of freedom with Fermi statistics. The classical gauge transformations are not appropriate for quantum structures, they can be used only for noninteracting particle degrees of freedom. The indefinite unitary time representations with eigenvectors and nilvectors will be shortly considered in this section, first in an algebraic matrix model, then in the associated quantum algebras and, finally, in the relativistic gauge fields.

5.1. Indefinite Unitarity for Ghosts

The dynamics of a Newtonian free mass point with Hamiltonian $H = \frac{p^2}{2M}$

$$\begin{pmatrix} x \\ ip \end{pmatrix}(t) = \begin{pmatrix} 1 & -i\frac{t}{M} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ ip \end{pmatrix} \iff \begin{cases} x(t) = x(0) + \frac{t}{M}p(0) \\ p(t) = p(0) \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ ip \end{pmatrix} = i\begin{pmatrix} 0 & -\frac{1}{M} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ ip \end{pmatrix}$$

[6] Ghosts, as they are called in physics, are nontrivial vectors with trivial norm (inner product). They can occur only for a nontrivial signature (indefinite metric). Mathematicians call this property isotropic. Ghost come in pairs, characterized by the minimal indefinite groups $\mathbf{O}(1,1)$ or $\mathbf{U}(1,1)$.

is a faithful nondiagonalizable time representation, in general

$$e^t \mapsto e^{iEt}\begin{pmatrix} 1 & i\nu t \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad E, \nu \in \mathbb{R}, \nu \neq 0$$

$$t \mapsto i\begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix}^t$$

For the Newtonian mass point momentum is an eigenvector with trivial time translation eigenvalue $E = 0$, position is a nilvector (principal vector, no eigenvector).

The Hamiltonian matrix is **U**(1,1)-Hermitian and the group representation **U**(1,1)-unitary (section 2)

$$H = \begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix} = H^\times$$

$$D(t) = e^{iEt}\begin{pmatrix} 1 & i\nu t \\ 0 & 1 \end{pmatrix} = D(-t)^\times$$

The Hamilton-matrix is the sum of two commuting transformations, the semisimple and the nilpotent part, called nil-Hamiltonian

$$H = E\mathbf{1}_2 + \nu N,\, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [H, N] = 0,\, N^2 = 0$$

The representation space cannot be spanned by energy eigenvectors alone which are characterized by the trivial action of the nil-Hamiltonian[7] eigenvector ("good")

$$|G\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle G| = (1, 0), \begin{cases} H|G\rangle = E|G\rangle, \quad N|G\rangle = 0 \\ (H - E)|G = 0 \\ |G\rangle(t) = e^{iEt}|G\rangle \end{cases}$$

In addition to the eigenvector one needs another principal vector

nilvector ("bad")

$$|B\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle B| = (0, 1), \begin{cases} H|B\rangle = E|B\rangle + \nu|G\rangle,\, N|B = \nu|G\rangle \\ (H - E)^2|B\rangle = 0 \\ |B\rangle(t) = e^{iEt}|B\rangle + i\nu t e^{iEt}|G\rangle \end{cases}$$

For the **U**(1,1)-inner product both eigenvectors and nilvectors are ghosts, i.e., their **U**(1,1)-norm vanishes

$$\mathbf{U}(1,1) : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} \langle G|G\rangle = 0,\ \langle G|B\rangle = 1 \\ \langle B|G\rangle = 1,\ \langle B|B\rangle = 0 \end{cases}$$

$$H = E(|B\rangle\langle G| + |G\rangle\langle B|) + \nu|G\rangle\langle G|$$

[7] Ghosts which are time translation eigenvectors are called "good ghosts"—"bad ghosts" are no time translation eigenvectors.

Eigen- and nilvectors form a not $\mathbf{U}(1,1)$-orthogonal Witt basis. The inner product gives no Hilbert space structure with the signature explicitly seen in Sylvester bases

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{cases} \langle G+B|G+B\rangle = 2, & \langle G+B|G-B\rangle = 0 \\ \langle G-B|G+B\rangle = 0, & \langle G-B|G-B\rangle = -2 \end{cases}$$

5.2. Ghosts in Quantum Structures

The quantum structure of the $\mathbf{U}(1,1)$-time representation in the last subsection becomes somewhat more complicated since the nilpotency of the nil-Hamiltonian $N^2 = 0$ for the matrix product (endomorphism product) is not implemented by the quantum algebra product using Bose degrees of freedom alone. This requires the introduction of additional twin-like Fermi degrees of freedom as done with the Fadeev–Popov (Kugo and Qjima, 1978) fields as Fermi twins for the nonparticle degrees of freedom in the Bose gauge fields.

The basic representation space is doubled. Therewith a $\mathbb{Z}_2$-graded quantum algebra (Section 4) is constructed as product of a Bose quantum algebra (capital letters G,B) and a Fermi quantum algebra (small letters g, b)

$$\mathbf{Q}_-(\mathbb{C}^4) \otimes \mathbf{Q}_+(\mathbb{C}^4) \cong \mathbb{C}[\mathrm{B},\mathrm{G},\mathrm{B}^\times, \mathrm{G}^\times] \otimes \mathbb{C}[b, g, \mathrm{b}^\times \mathrm{g}^\times]$$

$$\text{with} \quad \begin{cases} [\mathrm{G}^\times, \mathrm{B}] = 1, & [\mathrm{B}^\times, \mathrm{G}] = 1 \\ \{\mathrm{g}^\times, \mathrm{b}\} = 1, & \{\mathrm{b}^\times, \mathrm{g}\} = 1 \end{cases}$$

On the doubled quantum algebra the time development is implemented by the adjoint action of teh correspondingly doubled Hamiltonian

$$\mathbf{H}_{BF} = \mathbf{H}_B + \mathbf{H}_F, \quad \begin{cases} \mathbf{H}_B = E\frac{\{\mathrm{B},\mathrm{G}^\times\}+\{\mathrm{G},\mathrm{B}^\times\}}{2} + \nu \mathrm{G}\mathrm{G}^\times \\ \mathbf{H}_F = E\frac{[\mathrm{b},\mathrm{g}^\times]+[\mathrm{g},\mathrm{b}^\times]}{2} + \nu \mathrm{g}\mathrm{g}^\times \end{cases}$$

By products of Bose with Fermi operators a nilquadratic BRS-operator (Becehi, *et al.*, 1971; Saller, 1991, 1992, 1993a) of Fermi type can be constructed

$$\mathbf{N}_{BF} = \mathrm{g}\mathrm{G}^\times + \mathrm{G}\mathrm{g}^\times \Rightarrow [\mathbf{H}_{BF}, \mathbf{N}_{BF}] = 0, \mathbf{N}_{BF}^2 = 0$$

The quantum product $\mathbf{N}_B = \mathrm{G}\mathrm{G}^\times$ of Bose operators is not nilpotent.

On the doubled basic vector space the Hamiltonian matrix shows the block-diagonal doubling, the BRS matrix is skew-block-diagonal

$$H_{BF} = H_B \oplus H_F = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \left(\begin{array}{cc|cc} E & \nu & 0 & 0 \\ 0 & E & 0 & 0 \\ \hline 0 & 0 & E & \nu \\ 0 & 0 & 0 & E \end{array}\right)$$

$$N_{BF} = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The quantum BRS-operator effects—in analogy to the action of the nil-Hamiltonian action in the 2×2-matrix formulation—the projection to a subspace, spanned by time translation eigenvectors. The graded adjoint action of the BRS-operator

$$\text{ad}\,\mathbf{N}_{BF}(a) = \begin{cases} [\mathbf{N}_{BF}, a] \text{ for } a \text{ Bose}, & \text{e.g. } [\mathbf{N}_{BF}, \mathrm{G}] = 0 \\ \{\mathbf{N}_{BF}, a\} \text{ for } a \text{ Fermi}, & \text{e.g. } \{\mathbf{N}_{BF}, \mathrm{g}\} = 0 \end{cases}$$

defines the unital subalgebra of the doubled quantum algebra with the linear combinations of the time translation eigenvectors

$$\text{INV}_{\mathbf{N}_{BF}} \mathbf{Q}_{\pm}(\mathbb{C}^8) = \{p \in \mathbf{Q}_{-}(\mathbb{C}^4) \otimes \mathbf{Q}_{+}(\mathbb{C}^4) | \quad \text{ad}\,\mathbf{N}_{BF}(p) = 0\}$$

The product Fock space for $\mathbf{Q}_{-}(\mathbb{C}^4) \otimes \mathbf{Q}_{+}(\mathbb{C}^4)$, as constructed for the $\mathbf{U}(1)$-time representations in Section 4, has an indefinite metric

$$\text{FOCK}_{-}(\mathbb{C}^2) \otimes \text{FOCK}_{+}(\mathbb{C}^2) \quad \text{with} \quad \begin{cases} \langle \mathrm{G} \pm \mathrm{B} | \mathrm{G} \pm \mathrm{B} \rangle = \pm 2 \\ \langle \mathrm{g} \pm \mathrm{b} | \mathrm{g} \pm \mathrm{b} \rangle \quad = \pm 2 \end{cases}$$

It cannot be used for a probability interpretation. The subspace with the time translation eigenvectors—i.e., the classes for the BRS-invariance algebra $\text{INV}_{\mathbf{N}_{BF}} \mathbf{Q}_{\pm}(\mathbb{C}^8)$ above

$$\{|p\rangle \in \text{FOCK}_{-}(\mathbb{C}^2) \otimes \text{FOCK}_{+}(\mathbb{C}^2) | \mathbf{N}_{BF} |p\rangle = 0\}$$

contains—up to $|0\rangle$ (the class of the quantum algebra unit 1) with $\langle 0|0\rangle = 1$—only normless vectors (ghosts), e.g., $\langle \mathrm{g}|\mathrm{g}\rangle = 0 = \langle \mathrm{G}|\mathrm{G}\rangle$. Its metric is semidefinite (pre-Hilbert space). The canonically associated Hilbert space with the definite classes is 1-dimensional $\mathbb{C}|0\rangle$, i.e. it contains only the classes of the scalars. From the whole operator quantum algebra for the $\mathbf{U}(1,1)$-time representation there is—apart form the scalars $\mathbb{C}|0\rangle$—no vector left for the asymptotic Hilbert space—ghosts have no asymptotic states, i.e. they have no particle projections.

5.3. Ghosts in Gauge Theories

$\mathbf{U}(1,1)$-time representations with the characteristic ghost pairs (B, G) arise in gauge fields. The Lorentz group compatible space–time translation representations for Lorentz vectors are in the indefinite unitary group $\mathbf{U}(1,3) \supset \mathbf{SO}_0(1,3)$. Orthogonal time–space bases (Sylvester bases) and lightlike bases (Witt bases) reflect the two bases for eigen- and nilvector used above

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_0^2 - x_3^2 = (x_0 + x_3)(x_0 - x_3)$$

The algebraic concepts used in the (2×2)-matrix language above are the left hand side of the following dictionary to the familiar gauge field language

$$\begin{aligned}
\text{nilconstant } \nu \neq 0 &\sim \text{gauge fixing constant} \\
\text{nil-Hamiltonian } N \text{ with } N^2 = 0 &\sim \text{Becchi–Rouet–Stora charge} \\
\text{nil-Hamiltonian action, e.g. } N|B\rangle = \nu|\mathrm{G}\rangle &\sim \text{gauge transformation} \\
\text{time translation eigenvectors, e.g. } N|\mathrm{G}\rangle = 0 &\sim \text{gauge invariant vectors} \\
\text{eigenvectors with nontrivial norm} &\sim \text{asymptotic particles} \\
\text{ghost pairs with trivial norm} &\sim \text{interaction without particles}
\end{aligned}$$

For a free massless electromagnetic gauge field the quantization

$$[\mathbf{A}^k, \mathbf{A}^j](x) = \int \frac{d^4q}{(2\pi)^3} \epsilon(q_0) \left[-\eta^{kj} + 2\nu q^k q^j \frac{\partial}{\partial q^2} \right] \delta(q^2) e^{iqx}$$

is contrasted with the quantization of a free massive vector field

$$[\mathbf{Z}^k, \mathbf{Z}^j](x) = \int \frac{d^4q}{(2\pi)^3} \epsilon(q_0) \left[-\eta^{kj} + \frac{q^k q^j}{q^2} \right] \delta(q^2 - m^2) e^{iqx}, \quad m^2 > 0$$

The gauge field employs the characteristic Dirac function derivative $\delta\prime(q^2)$, multiplied with the gauge fixing constant ν.

The harmonic analysis of the massive vector field with respect to the time representations with spin 1 involves the Lorentz transformation $\Lambda(\frac{q}{m})$ to a rest system with $\mathbf{SO}(3)$ the "little group" for energy–momenta with $q^2 = m^2 > 0$

$$\mathbf{Z}^j(x) = \int \frac{d^3q}{(2\pi)^3 q_0} \Lambda\left(\frac{q}{m}\right)^j_a \frac{\mathrm{u}^{\mathrm{a}}(\vec{q}) e^{iqx} + \mathrm{u}^{\star\mathrm{a}}(\vec{q}) e^{-iqx}}{\sqrt{2}} \quad \text{with} \quad q_0 = \sqrt{m^2 + \vec{q}^2}$$

$$\text{and } \Lambda\left(\frac{q}{m}\right) = \frac{1}{m} \left(\begin{array}{c|c} q_0 & \vec{q} \\ \vec{q} & \delta_{ab} m + \frac{q_a q_b}{m + q_0} \end{array} \right) \in \mathbf{SO}_0(1,3), \quad \Lambda\left(\frac{q}{m}\right) \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = q,$$

The time representation by the gauge fields

$$\mathbf{A}^j(x) = \int \frac{d^3q}{(2\pi)^3 q_0} h\left(\frac{\vec{q}}{|\vec{q}|}\right)^j \begin{pmatrix} \frac{[\mathrm{B}(\vec{q}) - i\nu x_0 q_0 \mathrm{G}(\vec{q})] e^{iqx} + (1+\nu) \mathrm{G}^{\times}(\vec{q}) e^{-iqx}}{\sqrt{2}} \\ \frac{\mathrm{u}^1(\vec{q}) e^{iqx} + \mathrm{u}_1^{\star}(\vec{q}) e^{-iqx}}{\sqrt{2}} \\ \frac{\mathrm{u}^2(\vec{q}) e^{iqx} + \mathrm{u}_2^{\star}(\vec{q}) e^{-iqx}}{\sqrt{2}} \\ \frac{(1+\nu) \mathrm{G}(\vec{q}) e^{iqx} + \left[\mathrm{B}^{\times}(\vec{q}) + i\nu x_0 q_0 \mathrm{G}^{\times}(\vec{q})\right] e^{-iqx}}{\sqrt{2}} \end{pmatrix}$$

with $\quad q_0 = |\vec{q}|$

involves—in addition to the two photonic particle degrees of freedom in the 1st and 2nd component with two-time representations in **U**(1)—the Coulomb interaction and gauge degrees of freedom in the 0th and 3rd component. $h(\frac{\vec{q}}{|\vec{q}|})$ is a representative of $\mathbf{SO}_0$(1,3)/**SO**(2) to transform to the polarization group **SO**(2) ("little group" for energy–momenta with $q^2 = 0, q \neq 0$)

$$h\left(\frac{\vec{q}}{|\vec{q}|}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & O(\frac{\vec{q}}{|\vec{q}|}) \end{array}\right) \circ w, \quad w = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c|c} 1 & 0 & -1 \\ \hline 0 & \mathbf{1}_2\sqrt{2} & 0 \\ \hline 1 & 0 & 1 \end{array}\right)$$

$$O\left(\frac{\vec{q}}{|\vec{q}|}\right) \in \mathbf{SO}(3) \quad \text{with} \quad O\left(\frac{\vec{q}}{|\vec{q}|}\right) \begin{pmatrix} 0 \\ 0 \\ |\vec{q}| \end{pmatrix} = \vec{q}$$

The **U**(1,1)-time representation structure of the additional Faddev-Popov scalar fields doubles the gauge and Coulomb degree of freedom {G,B} on the Fermi sector {g,b} as shown in the algebraic scheme of the former subsection and will not be given here explicitly.

The inner product structure can be seen in the decomposition of the indefinite time representation containing unitary group **U**(1,3) extending the Lorentz group—for massive vector fields with Sylvester bases

$$\left.\begin{array}{l} \text{massive particles,} \\ \text{e.g. stable} Z: \end{array}\right\} \quad \text{with} \quad \left\{\begin{array}{l} \mathbf{SO}_0(1,3) \hookrightarrow \mathbf{U}(1,3) \supset \mathbf{U}(1) \times \mathbf{U}(3) \\ \text{Lorentz metric} : \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \mathbf{1}_3 \end{array}\right) = -\eta \end{array}\right.$$

and for massless gauge fields with Witt bases

$$\left.\begin{array}{l} \text{massive ghosts} \\ \text{and particles, e.g.} \\ \text{photons} \gamma: \end{array}\right\} \quad \text{with} \quad \left\{\begin{array}{l} \mathbf{SO}_0(1,3) \hookrightarrow \mathbf{U}(1,3) \supset \mathbf{U}(1,1) \times \mathbf{U}(2) \\ \text{Lorentz metric} : \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & \mathbf{1}_2 & 0 \\ \hline 1 & 0 & 0 \end{array}\right) = -w^T \circ \eta \circ w \end{array}\right.$$

6. UNSTABLE STATES AND PARTICLES

Decaying states (particles) can be considered in a Hilbert space where they form, together with other states—stable or unstable—multidimensional probability collectives. Unstable particles lead to the consideration of non–unitary automorphisms of Hilbert spaces. The 2-dimensional neutral kaon system with the short- and long-lived unstable neutral kaon as an illustration suggests a more general algebraic formulation.

6.1. The Neutral Kaons as a Probability Collective

The system of the two neutral K-meson states $|K_{S,L}\rangle$ shows—on the one hand—the phenomenon of CP-violation and—on the other hand—is unstable and decays into many channels (treated with the general formalism in the next subsection).

The kaon particles are no CP-eigenstates $|K_\pm\rangle$ to which they can be transformed by an invertible (2×2)-matrix

$$\begin{pmatrix} |K_S\rangle \\ |K_L\rangle \end{pmatrix} = T \begin{pmatrix} |K_+\rangle \\ |K_-\rangle \end{pmatrix}, \quad T \in \mathbf{GL}(\mathbb{C}^2)$$

Under the assumption of CPT-invariance the matrix is symmetric and parameterizable by two complex numbers wherefrom—with irrelevant $\mathbf{U}(1)$-phases—the normalization N_K can be chosen to be real

$$T = T^T = \frac{1}{N_K\sqrt{1+|\epsilon|^2}} \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}, \quad \epsilon \in \mathbb{C}, \quad N_K \in \mathbb{R}$$

The time development is implemented by a Hamiltonian, non-Hermitian for the unstable states $H_K \neq H_K^\star$

$$\text{for} \quad t \geq 0 : \frac{d}{dt}\begin{pmatrix} |K_+\rangle \\ |K_-\rangle \end{pmatrix} = iH_K \begin{pmatrix} |K_+\rangle \\ |K_-\rangle \end{pmatrix},$$
$$\frac{d}{dt}\begin{pmatrix} |K_S\rangle \\ |K_L\rangle \end{pmatrix} = i\,\text{diag}\, H_K \begin{pmatrix} |K_S\rangle \\ |K_L\rangle \end{pmatrix}$$

with the diagonal form for the energy eigenstates

$$\text{diag} \quad H_K = \begin{pmatrix} M_S & 0 \\ 0 & M_L \end{pmatrix}, \quad M = m + i\frac{\Gamma}{2},\ m,\ \Gamma > 0$$
$$H_K = T^{-1} \quad \text{diag} \quad H_K T = \begin{pmatrix} M_S - \varepsilon^2 M_L & \varepsilon(M_S - M_L) \\ \varepsilon(M_L - M_S) & M_L - \varepsilon^2 M_S \end{pmatrix}$$

For the scalar product, constructed for $t = 0$, the CP-eigenstates constitute an orthogonal basis in a complex 2-dimensional Hilbert space

$$\text{CP-eigenstates:} \quad \begin{pmatrix} \langle K_+|K_+\rangle & \langle K_+|K_-\rangle \\ \langle K_-|K_+\rangle & \langle K_-|K_-\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

whereform there arises the positive[8] non-diagonal metrical matrix $T^\star T$ for the time translation eigenstates

$$\text{energy eigenstates:} \quad \begin{pmatrix} \langle K_S|K_S\rangle & \langle K_S|K_L\rangle \\ \langle K_L|K_S\rangle & \langle K_L|K_L\rangle \end{pmatrix} = T^\star T = \frac{1}{N_K^2}\begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}$$

[8] Any matrix product $f^\star f$ is unitarily equivalent to a positive diagonal matrix.

The experiments give a nontrivial transition between the short and long lived kaon—the real part of ϵ. Therefore T is not definite unitary

$$\delta = \frac{\epsilon + \overline{\epsilon}}{1 + |\epsilon|^2} \sim 0.327 \times 10^{-2} \Rightarrow T \notin \mathbf{U}(2)$$

The normalization $N_K = 1$, usually chosen, normalizes individually the particle states, i.e., $\langle K_S|K_S\rangle = \langle K_L|K_L\rangle = 1$. However, the kaon system is a 2-dimensional probability collective, i.e., a complex 2-dimensional Hilbert space. Therefore it can also be collectively normalized (different form the individual normalization) via the discriminant (determinant)

$$\begin{aligned}
\det \quad T^\star T = |\det T|^2 &= \langle \det T|\det T\rangle \\
&= \langle K_S|K_S\rangle\langle K_L|K_L\rangle - |\langle K_S|K_L\rangle|^2 = 1 \\
&= N_K^2 = \frac{\sqrt{(1-\epsilon^2)(1-\overline{\epsilon^2})}}{1+|\epsilon|^2} = 1 - \delta^2
\end{aligned}$$

A decomposition of the unit can be written with orthogonal CP-eigenvectors and nonorthogonal time translation eigenvectors (particle)

$$\begin{aligned}
\mathbf{1}_2 &= |K_+\rangle\langle K_+| + |K_-\rangle\langle K_-| \\
&= |K_S\rangle\langle K_S^\perp| + |K_L\rangle\langle K_L^\perp| = |K_S^\perp\rangle\langle K_S| + |K_L^\perp\rangle\langle K_L| \\
&= |K_S\rangle\langle K_S| - \delta|K_S\rangle\langle K_L| - \delta|K_L\rangle\langle K_S| + |K_L\rangle\langle K_L| \quad \text{for} \quad \frac{N_K^2}{1-\delta^2} = 1
\end{aligned}$$

6.2. Nonorthogonal Decaying States

The possibility to have nontrivial transition elements between particles, as $\langle K_S|K_L\rangle$ above, can be connected to the deviation from the definite unitary structures for unstable states. the following well-known theorems are relevant for the situation:

An $(n \times n)$-matrix H acting on a vector space $V \cong \mathbb{C}^n$, e.g. a Hamilton-matrix for the time translations, is unitarily equivalent to a diagonal matrix if, and only if it is normal—all concepts with respect to a definite $\mathbf{U}(n)$-conjugation

$$H = U \circ \operatorname{diag} H \circ U^\star \quad \text{with} \quad U \in \mathbf{U}(n) \iff H \circ H^\star = H^\star \circ H$$

In this case, the vector space can be decomposed into an orthogonal sum of the eigenspaces for N different eigenvalues $\operatorname{spec} H = \{M_k\}$

$$V = \bigoplus_{k=1}^{N} V_k, \quad \operatorname{diag} H = \bigoplus_{k=1}^{N} M_k \mathrm{id}_{V_k}, \quad \langle V_k|V_l\rangle = \{0\} \quad \text{for} \quad M_k \neq M_l$$

For $\mathbf{U}(n)$-Hermitian operators $H = H^\star$ the eigenvalues are real $M = E \in \mathbb{R}$.

An analogue (real) diagonal structure holds for (selfadjoint) normal operators on infinite dimensional Hilbert spaces.

Hamiltonians acting on a Hilbert space with complex eigenvalues $E + i\frac{\Gamma}{2}$, $\Gamma > 0$, have to be $\mathbf{U}(n)$-non-hermitian, $H \neq H^\star$. Only with at least one nonreal energy involved, i.e. one unstable particle, two time translation eigenvectors (particles) with different energies can have a nontrivial transition element—unstable particles have not to be orthogonal to other particles.

6.3. Probability Collectives for Decaying Particles

The structure with two translation eigenstates (particles) for unstable kaons can be generalized to q eigenstates $|M\rangle$ (particles) with the eigenvalues $M = m + i\frac{\Gamma}{2}$ involving at least one unstable state $\Gamma > 0$. An orthogonal basis, related to a particle basis $|M\rangle$ by a non–unitary transformation $T \notin \mathbf{U}$(q)is denoted by $|U\rangle$—generalizing the CP-eigenstates of the kaon collective. In addition all stable decay modes, assumed to be p translation eigenstates $|E\rangle$ (particles) with real eigenvalues E are included, e.g. $|\pi, \pi\rangle$, $|\pi, \pi, \pi\rangle$, $|\pi, l, \nu_l\rangle$ for the kaon collective

$$|M\rangle = \left(|m_j + i\frac{\Gamma_j}{2}\right)_{j=1}^{q} \quad \begin{cases}\text{mass eigenstates with at least}\\ \text{one decaying channel}\end{cases}$$

$$|U\rangle = \big(|U_j\rangle\big)_{j=1}^{q} \qquad \text{orthogonal states}$$

$$|E\rangle = (|E_l\rangle)_{l=1}^{p} \qquad \begin{cases}\text{stable eigenstates}\\ \text{(decay channels)}\end{cases}$$

In a more general formulation also a continuous momentum dependence can be included.

The translation eigenstates have the time development with a diagonal Hamiltonian

$$\text{for} \quad t \geq 0 : \frac{d}{dt}\begin{pmatrix}|U\rangle\\|E\rangle\end{pmatrix} = iH\begin{pmatrix}|U\rangle\\|E\rangle\end{pmatrix}, \quad WHW^{-1} = \text{diag}\, H = \begin{pmatrix}M & 0\\ 0 & E\end{pmatrix}$$

The non–unitary transformation $W \notin \mathbf{U}(q + p)$ is the product of a triangular matrix with a $(p \times q)$-matrix w for the decay from unstable particles to decay products, called Wigner–Weisskopf matrix (Wigner and Weisskopf, 1930), and a $(q \times q)$-matrix T for the transformation from unstable particles to orthogonal states (no particles)

$$W = \frac{1}{N}\begin{pmatrix}\mathbf{1}_q & w\\ 0 & \mathbf{1}_p\end{pmatrix}\begin{pmatrix}T & 0\\ 0 & \mathbf{1}_p\end{pmatrix} = \frac{1}{N}\begin{pmatrix}T & w\\ 0 & \mathbf{1}_p\end{pmatrix}$$

$$H = \begin{pmatrix} T^{-1} & 0 \\ 0 & \mathbf{1}_p \end{pmatrix} \begin{pmatrix} M & w(M-E) \\ 0 & E \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & \mathbf{1}_p \end{pmatrix}$$
$$= \begin{pmatrix} H_U & T^{-1}w(M-E) \\ 0 & E \end{pmatrix}$$

The mass eigenstates $|M\rangle$ have projections both on the orthogonal states and on the decay channels

$$\text{eigenstates:} \quad \begin{pmatrix} |M\rangle \\ |E\rangle \end{pmatrix} = \frac{1}{N} \begin{pmatrix} T & w \\ 0 & \mathbf{1}_p \end{pmatrix} \begin{pmatrix} |U\rangle \\ |E\rangle \end{pmatrix} = \frac{1}{N} \begin{pmatrix} T|U\rangle + w|E\rangle \\ |E\rangle \end{pmatrix}$$

The scalar product matrix for the probability collective with the $(q+p)$ translation eigenstates arises from the diagonal matrix with the orthogonal states and the decay channels

$$\begin{pmatrix} \langle U|U\rangle & \langle U|E\rangle \\ \langle E|U\rangle & \langle E|E\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{1_p} & 0 \\ 0 & \mathbf{1}_q \end{pmatrix},$$
$$\begin{pmatrix} \langle M|M\rangle & \langle M|E\rangle \\ \langle E|M\rangle & \langle E|E\rangle \end{pmatrix} = W^\star W = \frac{1}{N^2} \begin{pmatrix} T^\star T & T^\star w \\ w^\star T & \mathbf{1}_p + w^\star w \end{pmatrix}$$

with the collective normalization

$$\langle \det W|\det W\rangle = \langle M|M\rangle\langle E|E\rangle - \langle M|E\rangle\langle E|M\rangle$$
$$= \det T^\star T[\det(\mathbf{1}_p + w^\star w) - \det w^\star w] = N^2 = 1$$

The scalar product $\langle\,|\rangle$ of the complex space $V \cong \mathbb{C}^n$, $n = q + p$, with the translation eigenstates (particles) involving at least one unstable state is a positive matrix, different from the unit matrix. It can be factorized with a nonunitary matrix $W \notin \mathbf{U}(n)$ chosen as a representative of the orientation manifold $\mathbf{GL}(\mathbb{C}^n)/\mathbf{U}(n)$

$$\mathbf{1}_n \neq \langle\,|\,\rangle = W^\star W$$

The individual probability normalization for one state by the scalar product (Section 4)

$$\text{for} \quad \mathbf{U}(1) : \langle u|u\rangle = 1$$

is generalized to a collective normalization by the discriminant

$$\text{for} \quad \mathbf{U}(n) : \langle \det W|\det W\rangle = \det\langle\,|\,\rangle = 1$$

The invariance group of a scalar product in diagonal bases

$$\mathbf{U}(n) = \{U \in \mathbf{GL}(\mathbb{C}^n) | U^\star \mathbf{1}_n U = \mathbf{1}_n\}$$

is equivalent to the invariance group of the scalar product, reoriented by W in a particle basis to $W^\star \mathbf{1}_n W = \langle\,|\rangle$

$$\{G \in \mathbf{GL}(\mathbb{C}^n) | G^\star \langle\,|\,\rangle G = \langle\,|\,\rangle\} = W^{-1}\mathbf{U(n)W}$$

i.e. W is determined up to $\mathbf{U}(n)$.

A probability collective including decaying particles (translation eigenstates with complex energies), e.g., the two neutral kaons, together with their decay products, has a holistic identity. There exist many bases, e.g. nonorthogonal particle states and orthogonal CP-eigenstates for the kaon system. What can be measured are transition amplitudes between particles states. Via the nonvanishing transition elements (nonorthogonal) the identity of the energy eigenstates is spread over the whole collective. Obviously, for a small width $\frac{\Gamma}{m} << 1$ 1, the "uncomplete identity" of a decaying particle, i.e. its being part of a collective, will be difficult to discover. However, it is to be expected that there are experiments which can test the difference between an individual probability interpretation of decaying particles (1-dimensional Hilbert subspaces) and their collective probability interpretation (higher dimensional Hilbert subspaces) where, e.g., the difference between individual and collective discriminant normalization becomes relevant.

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